

Ultra-high energy collisions in static space-times: single versus multi-black hole cases

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We consider collision of two particles near static electrically charged extremal black holes and elucidate the conditions under which the energy in the centre of mass frame $E_{c.m.}$ grows unbounded. For a single black hole, we generalize the results obtained earlier for the Reissner-Nordström metric, to distorted black holes. In the multi-black hole space-time, qualitatively new features appear. If the point of collision is close to at least two horizons simultaneously, unbounded $E_{c.m.}$ are possible (i) without fine-tuning of particles' parameters, (ii) for an arbitrary mutual orientation of two velocities. Such a combination of properties (i) and (ii) has no analogues in the single black hole case and facilitates the condition of getting unbounded $E_{c.m.}$. Collisions in the electro-vacuum Majumdar-Papapetrou metric (several extremal black holes in equilibrium) is analyzed explicitly.

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I. INTRODUCTION

High energy processes near black holes are of important both from the theoretical and astrophysical points of view. Now, the interest to such processes increased significantly after findings of Bañados, Silk and West (hereafter, BSW). They found that the energy $E_{c.m.}$ in the centre of mass frame of two particles colliding near the extremal Kerr black hole, can grow

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unbounded [1]. Meanwhile, high energy collision near rotating black holes were considered before in a series of papers [2] - [4]. There is a crucial difference, however, between these papers and [1]. In the first case, unbounded $E_{c.m.}$ is achieved when both colliding particles move in the same direction towards a black hole. This is possible if one of colliding particles has fine-tuned parameters. In the situation discussed in [2] - [4], the effect is due to head-on collision and does not require any fine-tuning. (There is also an intermediate case when collision occurs on the near-horizon circular orbit.) However, another complication arises here. As one particle in the immediate vicinity of the horizon should move not towards a black hole but away from it, this requires special conditions - say, necessity of preceding collisions (multiple scattering scenario). In both situations [1] and [2] - [4], it is implied that a single black hole is present.

Quite recently, new set-up was suggested in [5]. In that paper, collisions in the multi-black hole metric were studied. More specifically, the exact Majumdar-Papapetrou solution of electrovacuum Einstein equations [10], [11] was exploited with the concrete analysis of geodesic particle motion in the background of two black holes. Unbounded $E_{c.m.}$ was obtained for extremely small space separation between black holes. In doing so, different cases were compared in [5] to the BSW effect near a single black hole depending on the sign of radial velocities (whether particles move in the same direction or in the opposite one).

The goal of the present work is to show that unbounded $E_{c.m.}$ can be achieved near multi-black holes for collision of *any* two particles moving in the vicinity of multi-black hole metric (starting from the two black hole configuration). The angle ψ between their velocity can be arbitrary, in the case $\psi = 0$ or $\psi = \pi$ we return to the situation considered in [5]. This degree of freedom has crucial consequences since, as will be clear below, it enables to arrange high-energy collision without fine-tuning typical of the BSW effect [6]. And, multiple scattering is not required now to have a particle moving away from a black hole. The fact that important constraints on getting such energies are relaxed, enlarges chances that the effects under discussion can have (at least, in principle) observational relevance.

We suggest a unified picture for the cases of multi-black and single black holes and give full classification of cases when unbounded $E_{c.m.}$ are possible for charged distorted extremal black holes.

The paper is organized as follows. In Sec. II we consider a generic single electrically charged extremal black hole without assumption of spherical symmetry. Equations of motion

are listed, the conservation law is formulated in terms of the spatial velocity. In Sec. III we analyze particle collisions in this background and classify the possible cases depending on whether or not unbounded $E_{c.m.}$ are possible. In Sec. IV we discuss the multi-black hole metric using the Majumdar - Papapetrou solution as an example. In Sec. V, general situation for the case of two black holes is discussed. Two kinds of limiting transitions in revealed in Sec. VI, brief comparison with previous results on this subject [5] is made. In Sec. VII, we discuss the problem of collisions from a general viewpoint analyzing kinematic underlying factors that lead to unbounded $E_{c.m.}$ Summary of main results is given in Sec. VIII.

Throughout the paper we use units in which fundamental constants are $G = c = 1$.

II. SINGLE BLACK HOLE: BASIC EQUATIONS

Let us consider the generic metric of a static black hole. It can be written in the Gauss normal coordinate system (that always exists at least in some vicinity of the horizon):

$$ds^2 = -N^2 dt^2 + dn^2 + g_{AB} dx^A dx^B, \quad (1)$$

where $A, B = 2, 3$ and all coefficients do not depend on t . The horizon lies at $N = 0$. Let us consider equations of motion for test particles. The energy $E = -P_t$ is conserved due to staticity. Here, P_μ is the generalized momentum. Let us suppose that the system is electrically charged. Then,

$$mu_t = P_t + q\varphi, \quad (2)$$

φ is the electric potential, q being the particle's charge, m the particle's mass, u^μ the four-velocity. The equations of motion read

$$m\dot{t} = \frac{X}{N^2}, \quad (3)$$

$$X = E - q\varphi, \quad (4)$$

$$m\dot{n} = \varepsilon \frac{Z}{N}, \quad (5)$$

$$Z = \sqrt{X^2 - N^2(m^2 + g_{AB}u^A u^B)}. \quad (6)$$

Dot denotes derivative with respect to the proper time τ . The parameter $\varepsilon = -1$ if a particle moves towards a black hole and $\varepsilon = +1$ in the opposite case. To derive (5), we used the

normalization condition

$$u_\mu u^\mu = -1. \quad (7)$$

To get insight into kinematics of motion, it is instructive to introduce the orthogonal tetrad basis $h_{(a)\mu}$ (a runs from 0 to 3) and define

$$V_{(i)} = V^{(i)} = -\frac{u^\mu h_{\mu(i)}}{u^\mu h_{\mu(0)}} \quad (8)$$

(its counterpart for rotating black holes is analyzed in Sec. III of [7]), $i = 1, 2, 3$.

For the metric (1), it is natural to introduce the tetrad according to

$$h_{(0)\mu} = -N(1, 0, 0, 0) \quad (9)$$

$$h_{(1)\mu} = (0, 1, 0, 0), \quad (10)$$

$$h_{(A)\mu} = (0, 0, s_{(A)2}, s_{(A)3}), \quad (11)$$

where $x^\mu = (t, n, x^2, x^3)$, $a = 2, 3$, $s_{(A)c} s_{(B)}^c = \delta_{AB}$, where $c = 2, 3$ and δ_{AB} is the Kronecker symbol. It follows from (3), (5), (8) that

$$V^{(1)} = \frac{\dot{n}}{\dot{t}N} = \varepsilon \frac{Z}{X}, \quad (12)$$

$$V^{(A)} = \frac{\dot{x}^b s_{(A)b} N}{X}. \quad (13)$$

One obtains from (12), (13) that

$$X = \frac{mN}{\sqrt{1 - V^2}} = mN\gamma_1. \quad (14)$$

Here,

$$V^2 = V^{(1)2} + V^{(2)2} + V^{(3)2}, \quad (15)$$

the individual gamma-factor

$$\gamma_1 = \frac{1}{\sqrt{1 - V_1^2}}, \quad (16)$$

$$\gamma_1 = \frac{X_1}{m_1 N}. \quad (17)$$

Eq. (14) coincides with eq. (29) of [8] but is valid in a more general situation, without assumption about spherical symmetry.

III. COLLISIONS NEAR SINGLE BLACK HOLE

Let us consider collision between two particles whose characteristics are labeled by indices 1 and 2. One can define the energy in the centre of mass frame $E_{c.m.}$ according to

$$E_{c.m.}^2 = -(m_1 u_1^\mu + m_2 u_2^\mu)(m_1 u_{1\mu} + m_2 u_{2\mu}) = m_1^2 + m_2^2 + 2m_1 m_2 \gamma, \quad (18)$$

where

$$\gamma = -u_{1\mu} u_2^\mu \quad (19)$$

has the meaning of the Lorentz factor of relative motion. Direct calculations gives us from (3), (5)

$$\gamma = \frac{X_1 X_2 - \varepsilon_1 \varepsilon_2 Z_1 Z_2}{m_1 m_2 N^2} - g_{AB} u_1^A u_2^B. \quad (20)$$

In general, we cannot solve equations of motion explicitly. Fortunately, this is not required in the given context. We only assume that $g_{AB} u_1^A u_2^B$ remains finite on the horizon. In the spherically symmetric space-time and for radial motion, $g_{AB} u_1^A u_2^B = 0$. For the Reissner-Nordström metric, we return to the case considered in [9].

Now, we want to analyze (20) thus generalizing the results [9] to the distorted black holes when spherical symmetry is not required. In what follows, we call a particle critical, if $X_H = 0$, and usual if $X_H \neq 0$. If $X_H \neq 0$ but is extremely small, we call a particle near-critical. Here, subscript "H" denotes the quantity calculated on the horizon. Actually, $X_H > 0$ for usual particles due to the forward in time condition $\dot{t} > 0$.

Let us consider the vicinity of the horizon. Then, for small N , we have for a usual particle the expansion

$$Z = X_H - \frac{N^2(m^2 + g_{AB} u^A u^B)_H}{2X_H} + O(N^4). \quad (21)$$

It is also seen from (12), (13) that in the horizon limit the velocity is directed along the normal to the horizon, so $V^{(3)} = O(N) \ll V^{(1)} \approx \varepsilon$. Thus for the angle ϕ_0 between the velocity and the normal to the horizon we have

$$\cos \phi_0 = \pm 1 \quad (22)$$

depending on the direction of motion.

From now on, we assume that black holes under considerations are extremal. This enables to avoid complications connected with the fact that the critical particle cannot reach the

horizon of a nonextremal black hole [6]. For the critical particle, assuming the validity of the Taylor expansion, we can write

$$X = CN + DN^2 + \dots \quad (23)$$

Then,

$$Z \approx N \sqrt{C^2 - m^2 - (g_{AB} u_1^A u_2^B)_H}. \quad (24)$$

It follows from (12), (13) that both components of the velocity $V^{(1)}$ and $V^{(3)}$ are, generally speaking, separated from zero and have the same order, so the angle between the velocity and the normal to the horizon ϕ_0 can be arbitrary depending on particle's characteristics.

Now, we will consider collision between particles in two cases separately.

A. Analogue of BSW effect, $\varepsilon_1 \varepsilon_2 = +1$

If collision occurs between two usual or two critical particles, it follows from (20) that γ remains finite on the horizon. The only case of interest is when, say, particle 1 is critical and particle 2 is usual. Then,

$$\gamma \approx \frac{(X_2)_H (C_1 - \sqrt{C_1^2 - (m_1^2 + g_{AB} u_1^A u_1^B)_H})}{N} \quad (25)$$

is unbounded that represents just the analogue of the BSW effect.

B. Head-on collisions, $\varepsilon_1 \varepsilon_2 = -1$

Now, two particles move in the opposite radial directions. For collisions of two usual particles near the horizon,

$$\gamma \approx \frac{2 (X_1)_H (X_2)_H}{m_1 m_2 N^2}. \quad (26)$$

Thus if N is small enough, γ can become as large as one likes.

If critical particle 1 collides with a usual one 2,

$$\gamma \approx \frac{2C_1 (X_2)_H}{m_1 m_2 N}. \quad (27)$$

The Lorentz factor grows more slowly but it diverges on the horizon anyway. For collision of two critical particles, γ remains finite.

Thus we can enumerate all possible configurations and the results for γ in Table 1. Here, "u -c" means collision between a usual and the critical particles, etc.

	opposite directions, $\varepsilon_1\varepsilon_2 = -1$	arbitrary different directions	coinciding directions, $\varepsilon_1\varepsilon_2 = +1$
$u - u$	N^{-2}	impossible	finite
$u - c$	N^{-1}	N^{-1}	N^{-1}
$c - c$	finite	finite	finite

Table 1. Possible cases of particle collisions near the horizon of a single black hole.

IV. MAJUMDAR-PAPAPETROU SYSTEMS

Now we will consider high energy collision in the background of the Majumdar - Papapetrou solution [10], [11]. Let we have n extremal black holes in equilibrium. Then,

$$ds^2 = -U^{-2}dt^2 + U^2(dx^2 + dy^2 + dz^2), \quad (28)$$

$$U = N^{-1} = 1 + \sum_{i=1}^n \frac{M_i}{l_i}, \quad l_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}, \quad (29)$$

M_i is a mass of i -th black hole. Each black hole has the electric charge $Q_i = M_i$. The "points" (x_i, y_i, z_i) are actually not the points in a usual sense but correspond to the horizons of a finite area [12].

The electrostatic potential

$$\varphi = 1 - U^{-1} = 1 - N, \quad (30)$$

where the constant is chosen to ensure $\varphi \rightarrow 0$ at infinity. It follows from (4) and (30) that

$$X_{1,2} = E_{1,2} - q_{1,2} + q_{1,2}N. \quad (31)$$

It is worth noting that, as the potential $\varphi = 1$ on each horizon, the criticality condition $X_H = 0$ reduces to $E = q$. Such a relation involves the characteristics of a particle itself only and is the same for all horizons. This simplifies greatly the analysis of particle collisions (see below). If, say, particle 1 is critical, we have

$$X_1 = \frac{q_1}{U} \quad (32)$$

that agrees with (23), where now only the first term is nonzero.

Now, the natural choice of a tetrad is slightly different from (9) - (11):

$$h_{(0)}^\mu = U(1, 0, 0, 0), \quad h_{(0)\mu} = -U^{-1}(1, 0, 0, 0), \quad (33)$$

$$h_{(1)}^\mu = U^{-1}(0, 1, 0, 0), \quad h_{(1)\mu} = U(0, 1, 0, 0) \quad (34)$$

and similarly for $h_{(2)}^\mu, h_{(3)}^\mu$, where $x^\mu = (t, x, y, z)$. Then, the tetrad components of a velocity

$$V_{(i)} = V^{(i)} = \frac{m\dot{x}^i}{X}, \quad (35)$$

$$V^2 = 1 - \frac{m^2}{X^2 U^2}. \quad (36)$$

If two particles collide, simple calculation of (19) gives us

$$\gamma = \gamma_1 \gamma_2 (1 - \vec{V}_1 \vec{V}_2), \quad \gamma_{1,2} = \frac{X_{1,2}}{m_{1,2}} U, \quad (37)$$

where the scalar product

$$\vec{V}_1 \vec{V}_2 = \sqrt{1 - \frac{m_1^2}{X_1^2 U^2}} \sqrt{1 - \frac{m_2^2}{X_2^2 U^2}} \cos \psi \quad (38)$$

is calculated in the flat Euclidean space, ψ being the angle between \vec{V}_1 and \vec{V}_2 in the point of collision.

The key moment consists in the appearance of the factor $\cos \psi$, where ψ is the angle between three-velocities of particles in the point of collision. In the case of a single black hole, any usual particle approaches the horizon along the normal to it according to (22), so $\cos \psi = \pm 1$. By contrast, now this a free parameter. This is quite natural since if there are several black holes, a particle cannot have a velocity that is perpendicular to all of them, even if separation between different black holes is small.

Let collision between two particles occur just in the region where separation between different black holes is small, so U is large, $N = U^{-1}$ is small. The analysis of γ goes similarly to the case of a single black hole and we obtain the following table of possible situations.

	γ for different directions, $\psi \neq 0$	γ for coinciding directions, $\psi = 0$
$u - u$	N^{-2}	finite
$u - c$	N^{-1}	N^{-1}
$c - c$	finite	finite

Table 2. Possible cases of particle collisions near multi-black hole for small separation.

It is seen from (38) that the relative sign of velocities is determined by $\cos \psi$. Let, say, two usual particles collide. Then, for $\psi = 0$ (particles move in the same direction) and $\psi = \pi$ (particles move in the opposite direction) the results presented in Table 2 reduce to those in Table 1 for $\varepsilon_1 \varepsilon_2 = +1$ and $\varepsilon_1 \varepsilon_2 = -1$, respectively.

Meanwhile, Table 2 contains some new variants to achieve unbound γ that were impossible according to Table 1. Now, this becomes possible if two usual particles or one usual and one critical particles move in different (in particular, opposite) directions.

If the point of collision is much more close to one of black holes than to others (say, to black hole 1), we can leave in the sum (29) the corresponding term only. Then, the problem is reduced to that considered in the previous Section devoted to a single black hole.

V. TWO BLACK HOLES: BASIC EQUATIONS

To illustrate the foregoing general features, let us consider the case of two black holes of equal masses M situated on the z -axis in the points $+a$ and $-a$. This is just the example discussed in [5]. We compare the approach of [5] with ours and reveal that, actually, there are two different kinds of limits that lead to high energy collisions.

It is instructive to make now transformation $x = \rho \cos \phi$, $y = \rho \sin \phi$ to the cylindric coordinate system in which

$$ds^2 = -U^{-2} dt^2 + U^2 (d\rho^2 + \rho^2 d\phi^2 + dz^2). \quad (39)$$

Here,

$$U = 1 + \frac{M}{\sqrt{\rho^2 + (z - a)^2}} + \frac{M}{\sqrt{\rho^2 + (z + a)^2}}. \quad (40)$$

In what follows, we restrict ourselves by motion in the equatorial plane $z = 0$, where

$$U = 1 + \frac{2M}{\sqrt{\rho^2 + a^2}}. \quad (41)$$

A. Motion with zero angular momenta

To begin with, we consider the case when both particles move along straight lines, so their angular momenta are equal to zero like in Sec. 4 of [5]. However, we make emphasis on non-collinear motion in this case that expands the set of possibilities how to obtain high γ and $E_{c.m.}$.

It is easy to show that, if a particle starts its motion along x-axis, it keeps moving along this direction, so $z = 0 = y$ [5]. However, if we rotate the axis at the angle ψ , nothing changes. Therefore, particle 2 can move along the direction $\phi = \psi = \text{const}$. We have from (7) and (3) that

$$\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 = \dot{\rho}^2 + \rho^2 \dot{\phi}^2 = \frac{X^2 - m^2 N^2}{m^2}. \quad (42)$$

Let two particle pass through the center $x = y = 0 = \rho$ and collide just in this point. Then, $\phi = 0$ for particle 1 and $\phi = \psi = \text{const}$ for particle 2. Calculating the gamma factor (19) and taking into account (37), (38), (42) one obtains that

$$\gamma = \frac{X_1 X_2 - Z_1 Z_2 \cos \psi}{m_1 m_2} U^2. \quad (43)$$

In the point of collision,

$$U = 1 + \frac{2M}{a}. \quad (44)$$

If $a \ll M$, we see that $U \gg 1$ and $N \ll 1$, so γ can become unbounded. The diversity of possibilities is just described by Table 2. Collisions studied in [5] correspond to $\psi = 0$, $\psi = \pi$.

Thus we do not need to analyze the full equations of motion that are quite cumbersome [5]. Moreover, actually there is no necessity to constraint motion by additional special conditions (like motion along the line). This is because there are general formulas (37), (38) that relate γ , characteristics of motion and the metric function U from which the effect of unbounded γ can be obtained if U is big enough.

B. More general set-up

Now, let a particle have the angular momentum L and move in the equatorial plane. Then, in the metric (39), its motion, by close analogy with (3) - (6), is described by equations

$$m\dot{t} = XU^2, \quad (45)$$

$$m\dot{\phi} = \frac{L}{U^2 \rho^2}, \quad (46)$$

$$m\dot{\rho} = \varepsilon ZU, \quad (47)$$

$$Z = \sqrt{X^2 - \frac{m^2}{U^2} - \frac{L^2}{U^4 \rho^2}}. \quad (48)$$

It is convenient to use the tetrads (now $x^i = t, \rho, z, \phi$)

$$h_{(0)\mu} = -U^{-1}(-1, 0, 0, 0), \quad (49)$$

$$h_{(1)\mu} = U(0, 1, 0, 0), \quad (50)$$

$$h_{(2)\mu} = U(0, 0, 1, 0), \quad (51)$$

$$h_{(2)\mu} = U(0, 0, 0, \rho). \quad (52)$$

The analogues of formulas (12), (13) read

$$V^{(1)} = V \cos \beta = \varepsilon \frac{Z}{X}, \quad (53)$$

$$V^{(3)} = V \sin \beta = \frac{L}{\rho U^2 X}, \quad (54)$$

where V is given by eq. (36), β is the angle characterizing direction of motion of an individual particle.

For the relative angle between particles in the point of collision we have

$$\cos \psi = (\varepsilon_1 \varepsilon_2 Z_1 Z_2 + \frac{L_1 L_2}{U^4 \rho^2}) \frac{1}{\sqrt{X_1^2 - m_1^2 U^{-2}}} \frac{1}{\sqrt{X_2^2 - m_2^2 U^{-2}}}. \quad (55)$$

Direct calculation of the Lorentz factor of relative motion of two particles (19) gives us

$$\gamma = \gamma_1 \gamma_2 (1 - \vec{V}_1 \vec{V}_2) = \frac{U^2}{m_1 m_2} [(X_1 X_2 - \varepsilon_1 \varepsilon_2 Z_1 Z_2) - \frac{L_1 L_2}{\rho^2 U^4}]. \quad (56)$$

Using (53), (54), (36), one can also write (56) in the form

$$\gamma = \frac{X_1 X_2 U^2}{m_1 m_2} (1 - \sqrt{1 - \frac{m_1^2}{X_1^2 U^2}} \sqrt{1 - \frac{m_2^2}{X_2^2 U^2}} \cos \psi). \quad (57)$$

If, say, $L_1 = 0$ from the very beginning, so $\sin \beta_1 = 0$, one can check using (48), (53) - (55) that eq. (56) is reduced to eq. (43).

VI. TWO KINDS OF LIMITING TRANSITIONS

We are interested in the behavior of γ for small separation a and small $\rho = \rho_0$ in the point of collision, where one can expect indefinitely large growth of γ . Correspondingly, there are two relevant limits depending on what quantity is sent to zero first.

A. Limit A, $\gamma_A \equiv \lim_{a \rightarrow 0} \lim_{\rho_0 \rightarrow 0} \gamma$

Let two particles collide in the centre or very closely to it. This means that in the point under consideration $\rho = \rho_0 \rightarrow 0$,

$$U^2 \rho_0 \rightarrow 0. \quad (58)$$

Then, the right hand side of (54) grows unbound both for the usual and critical particles, if $L \neq 0$. Meanwhile, the left hand side remains bounded, its value is less than 1. Therefore, for $L \neq 0$ the scenario under discussion is impossible. However, we can arrange this scenario, provided L becomes smaller and smaller in this process, $L \sim \rho_0 \rightarrow 0$. This applies to each of two particles and to the angle between them $\psi = \beta_1 - \beta_2$ that remains arbitrary. If separation a between two black holes is small, $U(\rho = 0) = 1 + \frac{2M}{a}$ is large. Then, we return to the case of collision for noncollinear motion considered above - see Table 2 and Subsection V A that describes now the behavior of γ_A .

B. Limit B, $\gamma_B \equiv \lim_{\rho_0 \rightarrow 0} \lim_{a \rightarrow 0} \gamma$

Now, instead of (58), the opposite case is realized:

$$U^2 \rho_0 \rightarrow \infty. \quad (59)$$

Now, L_1 and L_2 may be arbitrary. (But they cannot be zero simultaneously. This would correspond to collision in the centre, so we would return to case A instead of B.)

It follows from (54) that, for a usual particle, $\sin \beta \rightarrow 0$. Taking into account (41) and (32), one can see that for the critical particle β can be arbitrary. Therefore, the angle ψ between two usual particles is equal to 0 or π for two particles and can be arbitrary if at least one particle is critical.

This is also seen from (55). Let $Z_{1,2} \neq 0$, Then, for collision of two particles it follows from (48) that $Z \approx \sqrt{X^2 - \frac{m^2}{U^2}}$, the second term in (55) in parentheses is negligible as compared to the first one due to (59). We obtain

$$\cos \psi \approx \varepsilon_1 \varepsilon_2. \quad (60)$$

The Lorentz factor γ is finite if $\varepsilon_1 \varepsilon_2 = +1$ (motion in the same direction). If $\varepsilon_1 \varepsilon_2 = -1$ (motion in the opposite direction), this factor behaves like

$$\gamma \approx \frac{2X_1 X_2}{m_1 m_2} U^2 \quad (61)$$

and becomes unbound.

In a similar way, one can analyze the rest of possible configurations with the results that coincide with those presented in Table 1, provided $Z_{1,2} \neq 0$ in the point of collision.

C. Special case: collision near turning point

In the above treatment of eq. (55), the second term in parentheses was much smaller than the first one. The opposite situation arises if $Z_1(\rho_0) = 0$. Physically, it means that for particle 1, the collision point coincides with the turning point. Then, it is seen from (53) that $\cos \beta_1 = 0$. It follows from (48), (55), (56) that now

$$\cos \psi = \pm \frac{L_2}{U^2 \rho \sqrt{X_2^2 - m_2^2 U^{-2}}}, \quad (62)$$

$$\gamma = \frac{1}{m_1} \frac{1}{m_2} [X_1 X_2 U^2 - \frac{L_1 L_2}{\rho^2 U^2}]. \quad (63)$$

It is seen from (41) that ρU is finite in the limit under discussion. Correspondingly, it follows from (41) and (32) that particle 1 is critical, $X_1 \sim U^{-1}$. If particle 2 is also critical, $X_2 \sim U^{-1}$, so γ is finite, ψ can be arbitrary. If particle 2 is usual, $\cos \psi \rightarrow 0$ due to (59). Then, X_2 remains finite nonzero, $\gamma \sim U$ becomes unbound.

We see that the although collision in the turning point looks somewhat different from other cases, the results completely fall in the general scheme and are described by Table 1.

Formally, one can consider also limit C, for which

$$\rho = \alpha a \quad (64)$$

with $\alpha = O(1)$, so ρ_0 and a tend to zero with the equal rate. However, it is easy to check that limit C does not give any new as compared to limit B, so Table 1 applies here.

D. Comparison to previous studies

In Ref. [5] two situations were analyzed. In Sec. 4, collision with zero angular momenta is considered in the centre $\rho = 0$ that corresponds to our limit A. In this sense, our results give generalization to the case of an arbitrary angle ψ between particles. In Sec. 7, collision between particles with nonzero momenta was studied. Both particles were taken to be

identical, having angular momenta $L_{1,2} = \nu_{1,2}L$, where $\nu_{1,2} = \pm 1$, $L > 0$. Different configurations of collision, depending on $\varepsilon_1\varepsilon_2$ ($\beta_1\beta_2$ in notations of [5]) and $\nu_1\nu_2$ were analyzed in [5] in detail for different regions of parameters. Meanwhile, all of them are based on eq. (38) that coincides with our eq. (64). Therefore, they correspond to limit C that is equivalent to B, as is said above.

If $L = 0$, both $L_1 = L_2 = 0$. But this is impossible in case B or C, as is explained above. Although some equations of Sec. 7 of [5] do not contain L explicitly, L is contained there implicitly. Say, in eq. (67) of [5], L is actually present through the parameter σ defined in (63).

It is worth also noting that particles were assumed to be uncharged in [5], so both of them are usual. Our results represented in Table 2, include usual and critical particles.

E. Summary for collisions near two black holes

To summarize the results of the present Section, there are two different limits A and B (or, equivalently, C). For the corresponding regimes, two opposite relations (58) and (59) hold. In the first case, the angle ψ between particles is arbitrary, $L_{1,2} \rightarrow 0$. In the second one, momenta $L_{1,2}$ are arbitrary, the angle $\psi = 0$ or π for a usual particle or can be arbitrary for the critical one. It was stated in Sec. 8 that there is a crucial difference from the BSW effect near a single black hole. This is correct, but if one takes into account all possible processes near a single black hole (including motion not only towards a black hole), the whole set of possibilities for collisions near a single black hole and two black holes (in variant B or C) is the same. Therefore, case B (or C) does not give qualitatively new results from the high energy collisions near a single black hole, even in spite of the crucial difference in the geometries for such configurations.

Meanwhile, case A has no analogues for collisions near a single black hole at all.

Division to two different types of scenarios is connected with the high symmetry of a system due to which angular momenta of particle are preserved, there is a preferable direction of motion along the radius, etc. In a general case, when black holes are situated irregularly, one cannot expect analogues of a situation with a single hole in the region where gravitation fields of different holes overlap, so that scenarios of type B is not expected to be valid in general. Meanwhile, generalization of scenarios of type A seem to retain their validity since

they do not require any symmetry. This issue is discussed in the next Section.

VII. GENERAL KINEMATIC PICTURE

It is instructive to look at the problem from a more general viewpoint, not restricting ourselves by the metric (28). Say, we can include matter into consideration (dirty multi-black holes). Earlier, we showed that the growth of γ in the BSW effect, can be interpreted in terms of relative motion [8]. Below, we will relate directly γ to γ_1 and γ_2 characterizing motion of each particle, to extend consideration to a more general case of collision.

A. Flat space-time

First of all, let us consider collision of two particles 1 and 2 in the simplest case of the flat space-time. The laboratory frame is labeled by index 0. Taking, say, particle 1, we can use decomposition

$$u_1^\mu = \gamma_1 U^\mu + \beta_1 n^\mu, \quad (65)$$

where U^μ is the four velocity of an observer attached to the laboratory frame, n^μ is the vector orthogonal to it. One can obtain from (65) directly that

$$\gamma_1 = -U_\mu u_1^\mu. \quad (66)$$

It is seen that the quantity γ_1 has the meaning of the Lorentz factor of motion of particle 1 with respect to "particle" 0. In other words, this is just the gamma factor of particle 1 in the laboratory frame (individual gamma-factor). It follows from the normalization conditions $U_\mu U^\mu = u_\mu u^\mu = -1$ that

$$\beta_1^2 = \gamma_1^2 - 1. \quad (67)$$

Calculating (19), we obtain

$$\gamma = \gamma_1 \gamma_2 - \beta_1 \beta_2 \alpha. \quad (68)$$

Here, $\alpha = n_{1\mu} n^{2\mu}$, $|\alpha| < 1$.

In the laboratory frame,

$$u_a^\mu = \gamma_a(1, V_a \vec{n}_a), \quad \alpha = \vec{n}_1 \vec{n}_2 \equiv \cos \psi. \quad (69)$$

$$\alpha V_1 V_2 = \vec{V}_1 \vec{V}_2 \quad (70)$$

$$\gamma_{1,2} = \frac{1}{\sqrt{1 - V_{1,2}^2}}. \quad (71)$$

Eq. (68) can be rewritten as

$$\gamma = \frac{1}{\sqrt{1 - V^2}} = \gamma_1 \gamma_2 (1 - V_1 V_2 \cos \psi) = \gamma_1 \gamma_2 - a \sqrt{\gamma_1^2 - 1} \sqrt{\gamma_2^2 - 1}, \quad (72)$$

where V is the relative velocity

In a slightly different form,

$$\gamma = \gamma_1 \gamma_2 (1 - \vec{V}_1 \vec{V}_2). \quad (73)$$

Eqs. (72), (73) can be found, for example, in problem 1.3 of the problem book [13].

Now, one can enumerate all possible cases.

1) If both γ_1 and γ_2 are finite, γ is also finite irrespective of the sign of α .

2) Let $\gamma_1 \gg 1$, γ_2 is finite. Then,

$$\gamma \approx \gamma_1 (1 - \alpha \sqrt{\gamma_2^2 - 1}) = \gamma_1 (1 - \alpha V_2) \gg 1 \quad (74)$$

irrespective of the sign of α . It means that the relative velocity of particles, one of which moves with a speed separated from the speed of light and the other one almost with the speed of light, is always close to the speed of light.

3) Let $\gamma_1 \gg 1$, $\gamma_2 \gg 1$.

a) $\alpha \neq +1$. Then,

$$\gamma \approx \gamma_1 \gamma_2 (1 - \alpha) \gg 1 \quad (75)$$

is unbounded.

b) $\alpha = +1$.

$$\gamma \approx \frac{1}{2} \left(\frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right). \quad (76)$$

If $\gamma_1 \sim \gamma_2$, the gamma factor γ is finite. If $\gamma_1 \gg \gamma_2$ or vice versa, $\gamma \gg 1$.

B. Curved space-time

One can introduce the tetrad orthogonal basis. Then, the components of velocities should be understood according to (8). Previous formulas (71) - (73) retain their validity. For critical particle 1, γ_1 is finite. For usual particle 2, $\gamma_2 \sim N^{-1}$. There are two essential ingredients. (i) Collision occurs in the region where $N \ll 1$, (ii) this is achieved due to small separation between black holes. (ii) mutual orientation is arbitrary.

Let both particles be usual, so $V_1 \approx 1$ and $V_2 \approx 1$, Then, we have from (72) that

$$\gamma \approx \gamma_1 \gamma_2 (1 - \cos \psi) \quad (77)$$

is unbounded, provided $\psi \neq 0$. In doing so, $\gamma_1 \sim \gamma_2 \sim N^{-1}$, $\gamma \sim N^{-2}$.

If particle 1 is near-critical and particle 2 is usual,

$$\gamma \approx \gamma_1 \gamma_2 (1 - V_1 \cos \psi) \quad (78)$$

is also unbound. But now γ_1 is finite, $\gamma_2 \sim \gamma \sim N^{-1}$.

If both particles are near-critical, γ_1 and γ_2 are finite, there is no effect at all.

VIII. CONCLUSION

Thus we gave full classification of possible scenarios of collisions in the multi-black hole case. It unifies the BSW effect and head-on collisions in a more general coherent picture. We saw that high-energy collisions near black holes turn out to be "almost" universal phenomenon. There is a crucial difference between high energy collisions near a single black hole and those in the multi-black hole space-time. In the latter case, there is a free angle parameter that makes fine-tuning unnecessary. As a result, small N is compatible with arbitrary direction of the velocity. It is worth reminding that in the BSW effect, fine-tuning for one particle was mandatory [1], [6]. Now, the situation in a sense is opposite: it is seen from Table 2, that, rather, special conditions are required to avoid high-energy collisions!

It was already pointed out in Sec. IX of [5] that collisions with high $E_{c.m.}$ near two black holes can hint that a similar phenomenon should occur near more realistic spinning binary black holes. However, as without special symmetry the analysis of particle motion is too difficult, this remained as some hope. The results obtained in the present paper can be considered as partial confirmation of these hopes since we do not use details of equations of motion and rely on general kinematic reasonings. Therefore, the general approach under consideration seems to apply to the more realistic case of rotating black holes as well, although detailed separate analysis is desirable here.

IX. ACKNOWLEDGMENT

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